Journal of Statistical Physics, Vol. 58, Nos. 5/6, 1990

On the Bose Gas with Local Mean-Field Interaction

Manfred Schröder[1]

Received February 27, 1989; revision received September 13, 1989

A Bose gas model is considered where the interaction term is proportional to the integral over the square of the local particle density. This model exhibits a phase transition with the same critical behavior as the usual mean-field (imperfect) Bose gas, but only generalized condensation may occur, due to boundary conditions.

KEY WORDS: Bose gas; local mean-field interaction; boundary conditions; local particle density; ground state; generalized Bose condensation.

INTRODUCTION

As a very simple interacting Bose gas model the so-called imperfect or mean-field Bose gas has long been investigated[1-3] and can be said to be completely known and understood. But this model has one big disadvantage making it inappropriate for modeling real interacting systems: It ignores the spatial and energetic distribution of the particles. Therefore, several modifications of the mean-field model have been proposed and investigated, such as the Huang–Yang–Luttinger model[4-6] or the van der Waals limit of a fully interacting Bose gas, which gives, if there is no external potential, the same pressure as the mean-field model.[8,9]

The local mean-field model presented in this paper seems to be close to the van der Waals limit from the physical point of view: It gives the same scaled particle density for the ground state, even in the case of a scaled external potential, as can be shown by a simple variation ansatz after neglecting the kinetic energy in comparison with potential and interaction energy.

However, it requires different mathematical techniques: The local

[1] Sektion Mathematik, Karl-Marx-Universität Leipzig, Leipzig, DDR-7010, German Democratic Republic.

0022-4715/90/0300-1151$06.00/0

mean-field Hamiltonian is defined without a limiting procedure. Furthermore, since the system Hamiltonian commutes with the free Hamiltonian, it makes sense to compute expectation values of occupation numbers, particle densities, etc. In particular, it will be demonstrated that there may not occur Bose condensation (in the usual sense of macroscopic occupation of the single-particle ground state), but only generalized condensation in the sense defined in ref. 10.

Throughout this paper, I will consider, for the sake of simplicity of calculations, a Bose gas contained in a d-dimensional ($d \geqslant 3$) cubic box with Dirichlet boundary conditions on two opposite faces and periodic boundary conditions on the remaining surface. This can be interpreted as the model of a Bose gas enclosed between two hard walls at macroscopic distance.

The case of one attractive and one repulsive boundary instead of two Dirichlet boundaries, as described for the free Bose gas in ref. 11, and the effect of a scaled external potential (which gives a pressure different from that of the mean-field model) will be considered in a forthcoming paper.

1. DESCRIPTION OF THE MODEL

Given $L > 0$, denote $V_L = [-L/2, L/2]^{d-1} \times [0, L]$. Let h_L be the one-particle Hamiltonian, acting in the Hilbert space $\mathscr{H}_L = L_2(V_L)$. The Hamiltonian of the free Bose gas is $H_L^0 = d\Gamma(h_L)$, acting in the symmetric Fock space

$$\mathscr{F}_L^s = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \underbrace{\mathscr{H}_L \otimes_s \cdots \otimes_s \mathscr{H}_L}_{n \text{ times}}$$

Now let $\{e_{k,L}\}_{k \in K}$ be the set of eigenvalues of h_L corresponding to the normalized eigenfunctions $\{f_{k,L}\}_{k \in K} \subset \mathscr{H}_L$, and let $a(f)$, $a^+(f)$ denote the usual annihilation and creation operators in $\mathscr{F}_L^s$. We get

$$H_L^0 = \sum_{k \in K} e_{k,L} n_{k,L}$$

where $n_{k,L} = a^+(f_{k,L})\, a(f_{k,L})$ is the operator of the number of particles in the kth eigenstate.

The mean-field (imperfect) Bose gas is given by the Hamiltonian

$$\tilde{H}_L^a = H_L^0 + \frac{a}{2L^d} N_L^2, \qquad \text{with} \quad a > 0, \quad N_L = \sum_{k \in K} n_{k,L}$$

If we introduce the mean particle density operator $\rho_L = L^{-d} N_L$, we can rewrite the interaction term as $\frac{1}{2} a L^d \rho_L^2$.

In the following, we will consider a Bose gas with an interaction term behaving locally like the mean-field interaction: We introduce the local particle density operator

$$\rho_L(x) = \sum_{k \in K} |f_{k,L}(x)|^2 \, n_{k,L} \qquad \text{with} \quad x \in V_L$$

and define the Hamiltonian of the local mean-field Bose gas as

$$H_L^a = H_L^0 + \frac{a}{2} \int_{V_L} \rho_L^2(x) \, dx \tag{1}$$

If the Hamiltonian of one particle with mass m in the volume V_L is given by $h_L = -(\hbar^2/2m)\Delta$ with periodic or quasiperiodic boundary conditions, then automatically $|f_{k,L}(x)|^2 = L^{-d}$ for any k, x, and hence $H_L^a = \tilde{H}_L^a$. However, it does not remain true if other boundary conditions are posed.

In the present paper, we start from the one-particle Hamiltonian

$$h_L = -\frac{\hbar^2}{2m} \Delta \tag{2}$$

on V_L, with boundary conditions

$$\begin{aligned} &f|_{x_d=0} = f|_{x_d=L} = 0, \qquad f|_{x_l=-L/2} = f|_{x_l=L/2} \\ &\left.\frac{\partial f}{\partial x_l}\right|_{x_l=-L/2} = \left.\frac{\partial f}{\partial x_l}\right|_{x_l=L/2}, \qquad \text{for} \quad l = 1,..., d-1 \end{aligned} \tag{3}$$

Under this assumption, we obtain $K = \mathbb{Z}^{d-1} \times \mathbb{N}$, $e_{k,L} = (2\hbar^2\pi^2/mL^2) \times (k_1^2 + \cdots + k_{d-1}^2 + k_d^2/4)$,

$$f_{k,L}(x) = 2^{1/2} L^{-d/2} \exp\left[\frac{2\pi i}{L}(k_1 x_1 + \cdots + k_{d-1} x_{d-1})\right] \sin\left(\frac{\pi}{L} k_d x_d\right)$$

$$H_L^a = \sum_{k \in K} e_{k,L} n_{k,L} + \frac{a}{2L^d}\left(N_L^2 + \frac{1}{2}\sum_{j \in \mathbb{N}} N_{j,L}^2\right) \tag{4}$$

with

$$N_{j,L} = \sum_{\substack{k \in K \\ k_d = j}} n_{k,L}$$

2. THE GROUND STATE

The main advantage of the model in comparison with the usual mean-field model can be shown by considering the ground state of the system: Let $\rho = \langle \rho_L \rangle$ be the given mean particle density. The ground state of $\tilde{H}_L^a$ is established if all particles are in the state with index (0,..., 0, 1). This implies a scaled local particle density $\langle \rho_L(yL) \rangle = 2\rho \sin^2(\pi y_d)$, with $y \in V_1$. Obviously, such an essentially inhomogeneous particle distribution is untypical for real interacting systems. Now we will show that, in the local mean-field model, in the ground state, $\langle \rho_L(yL) \rangle \to \rho$ for $L \to \infty$ and $y \in V_1 \setminus \partial V_1$: Denote by σ_k the actual occupation number of the kth state at zero temperature, and let

$$s_j = \sum_{\substack{k \in K \\ k_d = j}} \sigma_k$$

Obviously, one has $\sigma_k = 0$ if $k' \neq 0$ [denoting here and in the sequel $k' = (k_1,..., k_{d-1})$]. Thus, it remains to solve the extremal value problem

$$\sum_{j=1}^{\infty} \frac{\hbar^2}{2m} \frac{\pi^2 j^2}{L^2} s_j + \frac{a}{4L^d} \sum_{j=1}^{\infty} s_j^2 = \text{minimal}$$

$$\sum_{j=1}^{\infty} s_j = \rho L^d$$

$$s_j \geqslant 0$$

It is easy to see that the sequence $\{s_j\}$ is monotonically decreasing and finite. Let $J = \min\{j: s_j = 0\}$. Now we solve the extremal value problem

$$\sum_{j=1}^{J} \frac{\hbar^2}{2m} \frac{\pi^2 j^2}{L^2} s_j + \frac{a}{4L^d} \sum_{j=1}^{J} s_j^2 - t\left(\sum_{j=1}^{J} s_j - \rho L^d\right) = \text{minimal}$$

where the Lagrange multiplier t can be found via the given total particle number, and J is to be chosen in a way that $s_J = 0$ is fulfilled. If L is large, then it will be sufficient to solve this problem for arbitrary real s_j (the resulting error in the local particle density will become small), which can be done explicitly. Hence we get approximately

$$J = \left(\frac{3amL^2\rho}{2\hbar^2\pi^2}\right)^{1/3}, \qquad t = \left(\frac{9a^2\hbar^2\pi^2\rho^2}{32mL^2}\right)^{1/3}$$

$$s_j = \begin{cases} \dfrac{2L^d}{a}\left(t - \dfrac{\hbar^2}{2m} \dfrac{\pi^2 j^2}{L^2}\right), & j < J \\ 0, & \text{else} \end{cases}$$

The result indicates that macroscopic occupation of any one-particle state does not occur (the occupation numbers are of order $L^{d-2/3}$).

Inserting now the result into the formula for the local particle density and replacing the sum by an integral, we obtain approximately

$$\langle \rho_L(x) \rangle = \rho(1 + 3q^{-2} \cos q - 3q^{-3} \sin q)$$

with

$$q = (12am\pi\rho/\hbar^2 L)^{1/3}\, x_d$$

For small q, the local density behaves asymptotically as $(\rho/10)\, q^2$. Since the characteristic length scale is of order $L^{1/3}$, we obtain $\langle \rho_L(yL) \rangle \to \rho$ as $L \to \infty$, for any $y \in V_1 \setminus \partial V_1$, which agrees with our intuition.

3. THE GRAND CANONICAL PRESSURE

Given the inverse temperature $\beta = (k_B T)^{-1}$, with k_B denoting the Boltzmann constant and T the temperature, and the chemical potential μ, one can evaluate the grand canonical pressure $p^a(\beta, \mu)$ via the grand canonical partition function

$$\mathscr{Z}_L^a(\beta, \mu) = \operatorname{Tr} \exp[-\beta(H_L^a - \mu N_L)]$$

as

$$p^a(\beta, \mu) = \lim_{L \to \infty} p_L^a(\beta, \mu)$$

where $p_L^a(\beta, \mu) = (\beta L^d)^{-1} \ln \mathscr{Z}_L^a(\beta, \mu)$. In this section, I will show that the pressure of the local mean-field Bose gas coincides with $\tilde{p}^a(\beta, \mu)$, the grand canonical pressure of the mean-field Bose gas, as derived in ref. 3.

Since the interaction term commutes with the free Hamiltonian, we can use the occupation number representation for the grand canonical partition function:

$$\mathscr{Z}_L^a(\beta, \mu) = \sum_{\sigma \in (\mathbb{Z}_+)^K} \exp\left\{ -\beta \left[\sum_{k \in K} (e_{k,L} - \mu)\, \sigma_k + \frac{a}{2L^d} \left(N^2 + \frac{1}{2} \sum_{j=1}^{\infty} s_j^2 \right) \right] \right\}$$

where the s_j are defined as in the previous section, and

$$N = \sum_{j=1}^{\infty} s_j$$

Analogously as in ref. 3, let us introduce an arbitrary (energy-valued) parameter $\alpha \leqslant 0$, and get

$$\mathscr{Z}_L^a(\beta, \mu) = \sum_{\sigma \in (\mathbb{Z}_+)^K} \exp\left\{-\beta\left[\frac{a}{2L^d}\left(N^2 + \frac{1}{2}\sum_{j=1}^{\infty} s_j^2\right) - (\mu - \alpha)N\right]\right\} \times \exp\left\{-\beta\left[\sum_{k \in K} (e_{k,L} - \alpha)\,\sigma_k\right]\right\} \tag{5}$$

On one hand, we have

$$\frac{a}{2L^d}\left(N^2 + \frac{1}{2}\sum_{j=1}^{\infty} s_j^2\right) - (\mu - \alpha)N \geqslant \frac{aN^2}{2L^d} - (\mu - \alpha)N \geqslant -\frac{L^d}{2a}(\mu - \alpha)^2$$

and hence we get, as outlined in ref. 3,

$$\mathscr{Z}_L^a(\beta, \mu) \leqslant \exp\left[\frac{\beta L^d}{2a}(\mu - \alpha)^2\right] \mathscr{Z}_L^0(\beta, \alpha)$$

which implies

$$\lim_{L \to \infty} \sup\, p_L^a(\beta, \mu) \leqslant p^0(\beta, \alpha) + \frac{(\mu - \alpha)^2}{2a} \tag{6}$$

Before I continue the calculation of the pressure, I introduce some notations:

$$F_L^{(d)}(E) = L^{-d}\,\mathrm{card}\{k \in K\colon e_{k,L} \leqslant E\}$$

$$F_L^{(d-1)}(E) = L^{-d+1}\,\mathrm{card}\left\{k' \in \mathbb{Z}^{d-1}\colon \frac{2\hbar^2\pi^2\,|k'|^2}{mL^2} \leqslant E\right\}$$

$$F^{(l)}(E) = \lim_{L \to \infty} F_L^{(l)}(E) = \frac{1}{\Gamma(l/2 + 1)}\left(\frac{mE}{2\hbar^2\pi}\right)^{l/2}, \qquad l = d-1, d \tag{7}$$

$$I_{r,L}^{(l)}(\beta, \mu) = \frac{1}{\beta^r}\frac{\partial^r}{\partial\mu^r}\int_0^{\infty} \ln\frac{1}{1 - \exp[-\beta(E - \mu)]}\, dF_L^{(l)}(E) \tag{8}$$

for $\mu \leqslant 0$; $r = 0, 1, 2$; $l = d-1, d$; and

$$I_r^{(l)}(\beta, \mu) = \lim_{L \to \infty} I_{r,L}^{(l)}(\beta, \mu) = \left(\frac{m}{2\beta\hbar^2\pi}\right)^{l/2} G_{l/2+1-r}[\exp(\beta\mu)]$$

for $\mu < 0$, where

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx \qquad (p > 0)$$

$$G_p(z) = \sum_{n=1}^\infty \frac{z^n}{n^p} \qquad (|z| < 1)$$

Finally, we set, for $\alpha \leqslant 0$, $t \in \mathbb{Z}_+$,

$$\Phi_{j,L}(\beta, \alpha, t) = \sum_{\substack{\{\sigma' \in (\mathbb{Z}_+)^{\mathbb{Z}^{d-1}}: \\ \sum_{k' \in \mathbb{Z}^{d-1}} \sigma'_{k'} = t\}}} \exp\left[-\beta \sum_{k' \in \mathbb{Z}^{d-1}} (e_{(k',j),L} - \alpha)\, \sigma'_{k'} \right] \tag{9}$$

Now it is easy to see that (5) can be rewritten as

$$\mathscr{Z}^a_L(\beta, \mu) = \exp\left[\frac{\beta L^d}{2a} (\mu - \alpha)^2 \right] \sum_{s \in (\mathbb{Z}_+)^{\mathbb{N}}} \prod_{j=1}^\infty \Phi_{j,L}(\beta, \alpha, s_j)$$
$$\times \exp\left[-\frac{\beta a}{2} L^d \left(\frac{N}{L^d} - \frac{\mu - \alpha}{a} \right)^2 - \frac{\beta a}{4L^d} \sum_{j=1}^\infty s_j^2 \right] \tag{10}$$

In order to estimate this sum from below, we diminish the summation range, leaving only a finite subset, and replace the items on the remaining index set by appropriate functions, for the purpose of getting a lower bound equal to that in (6) for a certain value of the parameter α.

Let $\delta \geqslant 0$, $x \geqslant 0$, $0 < \gamma < 1$, $0 < \lambda < 1$, $L > 0$, $J \in \mathbb{N}$, and define

$$M_j = \begin{cases} \left\{ \left[\dfrac{x}{\delta} L^{d-\gamma} \right] \right\} & \text{for} \quad 0 < j \leqslant \delta L^\gamma \\ [(1-\lambda)\, L^{d-1} I^{(d-1)}_{1,L}(\beta, \alpha - e_{(0,j),L}) - 1,\, (1+\lambda)\, L^{d-1} & \\ \quad \times I^{(d-1)}_{1,L}(\beta, \alpha - e_{(0,j),L}) + 1] \cap \mathbb{Z}_+ & \text{for} \quad \delta L^\gamma < j < J \\ \{0\} \quad \text{for} \quad j \geqslant J & \end{cases} \tag{11}$$

with $[t]$ denoting the integer part of t.

For $s \in \mathsf{X}_{j=1}^\infty M_j$ we obtain

$$(1-\lambda) \sum_{j=[\delta L^\gamma]}^{J} \frac{1}{L} I^{(d-1)}_{1,L}(\beta, \alpha - e_{(0,j),L}) + x + O(JL^{-d})$$
$$\leqslant \frac{N}{L^d}$$
$$\leqslant (1+\lambda) \sum_{j=[\delta L^\gamma]}^{J} \frac{1}{L} I^{(d-1)}_{1,L}(\beta, \alpha - e_{(0,j),L}) + x + O(JL^{-d})$$

which can be rewritten as

$$(1-\lambda)\, I_1^{(d)}(\beta, \alpha) + x + O(L^{\gamma-1} \ln L)$$
$$+ O\left(\left(\frac{J}{L}\right)^{d-1} e^{-\beta e_{(0,J),L}}\right) + O(JL^{-d})$$
$$\leqslant \frac{N}{L^d} \leqslant (1+\lambda)\, I_1^{(d)}(\beta, \alpha) + x + O(JL^{-d}) \tag{12}$$

Let us denote

$$R_1 = \sup\left\{\left(\frac{N}{L^d} - \frac{\mu-\alpha}{a}\right)^2 : s \in \mathop{\text{X}}_{j=1}^{\infty} M_j\right\} < \infty$$

Further, we get

$$\frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{s_j}{L^d}\right)^2 \leqslant \frac{1}{2}\frac{x^2}{\delta} L^{-\gamma} + \frac{1}{2}\sum_{j=[\delta L^\gamma]}^{J-1} (1+\lambda)^2 \frac{1}{L^2}$$
$$\times (I_{1,L}^{(d-1)}(\beta, \alpha - e_{(0,j),L}))^2$$
$$< \frac{1}{2}\frac{x^2}{\delta} L^{-\gamma} + \frac{1}{2L}(1+\lambda)^2 I_1^{(d)}(\beta, 0)$$
$$\times I_{1,L}^{(d-1)}\left(\beta, \alpha - \frac{\hbar^2\pi^2}{2m}\delta^2 L^{2\gamma-2}\right) + o(L^{-1}) =: R_2 \tag{13}$$

Hence we have

$$\mathscr{Z}_L^a(\beta, \mu) \geqslant \exp\left\{\frac{a\beta L^d}{2}\left[\left(\frac{\mu-\alpha}{a}\right)^2 - R_1 - R_2\right]\right\} \prod_{j=1}^{\infty} \sum_{s \in M_j} \Phi_{j,L}(\beta, \alpha, s)$$

resp.

$$p_L^a(\beta, \mu) \geqslant \frac{a}{2}\left[\left(\frac{\mu-\alpha}{a}\right)^2 - R_1 - R_2\right] + \frac{1}{\beta L^d}\sum_{j=1}^{\infty} \ln\left[\sum_{s \in M_j} \Phi_{j,L}(\beta, \alpha, s)\right]$$
$$= \frac{a}{2}\left[\left(\frac{\mu-\alpha}{a}\right)^2 - R_1 - R_2\right] + p_L^0(\beta, \alpha)$$
$$+ \frac{1}{\beta L^d}\sum_{j=1}^{\infty} \ln\frac{\sum_{s \in M_j} \Phi_{j,L}(\beta, \alpha, s)}{\sum_{s \in \mathbb{Z}_+} \Phi_{j,L}(\beta, \alpha, s)} \tag{14}$$

Now let us estimate the sum in (14). For this purpose, split it into four partial sums: $\sum_1$, with j running from 1 to $[\delta L^\gamma]$; $\sum_2$, with j from $[\delta L^\gamma] + 1$ to $J' - 1$; $\sum_3$, with j between J' and $J - 1$; and, finally, $\sum_4$, with $j \geqslant J$. The indices J, J' are defined in the following way:

$$J=\min\left\{j\in\mathbb{N}: L^{d-1}I_{1,L}^{(d-1)}(\beta,\alpha-e_{(0,j),L})\leqslant\frac{1}{2}\right\} \tag{15}$$

$$J'=\max\left\{j\in\mathbb{N}: \frac{I_{2,L}^{(d-1)}(\beta,\alpha-e_{(0,j),L})}{L^{d-1}\lambda^2[I_{1,L}^{(d-1)}(\beta,\alpha-e_{(0,j),L})]^2}\leqslant\frac{1}{2}\right\} \tag{16}$$

Appropriate estimations of $I_{r,L}^{(d-1)}$ by linear combinations of $I_r^{(l)}$, $l\leqslant d-1$, lead to the inequalities

$$\begin{aligned}&\frac{(2m)^{1/2}}{\hbar\pi}L\left[\alpha+\frac{d-1}{\beta}\ln(cL)\right]^{1/2}\\&\quad\leqslant J\\&\quad\leqslant\frac{(2m)^{1/2}}{\hbar\pi}L\left[\alpha+\frac{2}{\beta}\ln 2+\frac{d-1}{\beta}\ln(cL)\right]^{1/2}\end{aligned} \tag{17}$$

with $c=[m/(2\pi\beta\hbar^2)]^{1/2}$, and,

$$\begin{aligned}&\frac{(2m)^{1/2}}{\hbar\pi}L\left[\alpha+\frac{2}{\beta}\ln\frac{\lambda}{2}+\frac{d-1}{\beta}\ln(cL)\right]^{1/2}\\&\quad\leqslant J'\\&\quad\leqslant\frac{(2m)^{1/2}}{\hbar\pi}L\left[\alpha+\frac{2}{\beta}\ln\lambda+\frac{d-1}{\beta}\ln(cL)\right]^{1/2}\end{aligned} \tag{18}$$

for sufficiently large L.

Before we estimate $\sum_1$, $\sum_3$, and $\sum_4$, note that $\Phi_{j,L}(\beta,\alpha,s)\geqslant\exp[s\beta(\alpha-e_{(0,j),L})]$, and

$$\sum_{s\in\mathbb{Z}_+}\Phi_{j,L}(\beta,\alpha,s)=\exp[L^{d-1}I_{0,L}^{(d-1)}(\beta,\alpha-e_{(0,j),L})]$$

This implies

$$\begin{aligned}\sum\nolimits_1&\geqslant\sum_{j=1}^{[\delta L^\gamma]}\left\{\left[\frac{x}{\delta}L^{d-\gamma}\right]\beta(\alpha-e_{(0,j),L})\right.\\&\qquad\left.-L^{d-1}I_{0,L}^{(d-1)}(\beta,\alpha-e_{(0,j),L})\right\}\\&\geqslant(\delta L^\gamma)\left\{\frac{x}{\delta}L^{d-\gamma}\beta\left(\alpha-\frac{\hbar^2\pi^2}{2m}\delta^2L^{2\gamma-2}\right)-L^{d-1}I_{0,L}^{(d-1)}(\beta,\alpha)\right\}\\&=\beta\alpha xL^d-\frac{\beta\hbar^2\pi^2x\delta^2}{2m}L^{d+2\gamma-2}-\delta I_{0,L}^{(d-1)}(\beta,\alpha)\,L^{d-1+\gamma}\\&=\beta\alpha xL^d+o(L^d)\end{aligned} \tag{19}$$

Further, we have

$$\begin{aligned}
\Sigma_4 &= \sum_{j=J}^{\infty} \{-L^{d-1} I_{0,L}^{(d-1)}(\beta, \alpha - e_{(0,j),L})\} \\
&\geqslant \sum_{j=J}^{\infty} \{-L^{d-1} c^{d-1} \exp[\beta(\alpha - e_{(0,j),L})]\}[1 + O(L^{-1})] \\
&\geqslant -L^d c^{d-1}[1 + O(L^{-1})] \left\{ \int_{J/L}^{\infty} \exp\left[\beta\left(\alpha - \frac{\hbar^2\pi^2}{2m} y^2\right)\right] dy \right. \\
&\quad \left. + \frac{1}{L} \exp[\beta(\alpha - e_{(0,J),L})] \right\} \\
&= -L^d c^{d-1}[1 + O(L^{-1})] \left\{ \frac{(2m)^{1/2}}{\hbar\pi} e^{\beta\alpha} \int_{(\hbar^2\pi^2/2m)(J/L)^2}^{\infty} e^{-z} \frac{dz}{2\sqrt{z}} \right. \\
&\quad \left. + \frac{1}{L} \exp[\beta(\alpha - e_{(0,J),L})] \right\} \\
&\geqslant -L^d c^{d-1}[1 + O(L^{-1})] \left\{ \frac{(2m)^{1/2}}{\hbar\pi} \exp\left[\beta\alpha - \frac{\hbar^2\pi^2}{2m}\left(\frac{J}{L}\right)^2\right] \frac{m}{\hbar\pi(J/L)} \right. \\
&\quad \left. + \frac{1}{L} \exp[\beta(\alpha - e_{(0,J),L})] \right\} \\
&\geqslant -\left(\frac{\sqrt{2}\, m^{3/2}}{\hbar^2\pi^2 J} + 1\right)[1 + O(L^{-1})] = O(1)
\end{aligned} \tag{20}$$

(with c as above), due to (15), and

$$\begin{aligned}
\Sigma_3 &= \sum_{j=J'}^{J-1} \left\{ \ln\left[\sum_{s \in M_j} \Phi_{j,L}(\beta, \alpha, s)\right] \right. \\
&\quad \left. - L^{d-1} I_{0,L}^{(d-1)}(\beta, \alpha - e_{(0,j),L}) \right\} \\
&\geqslant (J - J')\{\beta(\alpha - e_{(0,J),L})(1 + \lambda)\, L^{d-1} I_{1,L}^{(d-1)}(\beta, \alpha - e_{(0,J),L}) \\
&\quad \times \ln[2\lambda L^{d-1} I_{1,L}^{(d-1)}(\beta, \alpha - e_{(0,J),L})] \\
&\quad - L^{d-1} I_{0,L}^{(d-1)}(\beta, \alpha - e_{(0,J'),L})\} \\
&= O(L(\ln L)^{1/2})
\end{aligned} \tag{21}$$

due to (15)–(18). Finally, note that

$$P_j(M) = \sum_{s \in M} \Phi_{j,L}(\beta, \alpha, s) \Big/ \sum_{s \in \mathbb{Z}_+} \Phi_{j,L}(\beta, \alpha, s)$$

is a probability measure on $\mathbb{Z}_+$ with expectation value $L^{d-1}I_{1,L}^{(d-1)}(\beta, \alpha - e_{(0,j),L})$ and variance $L^{d-1}I_{2,L}^{(d-1)}(\beta, \alpha - e_{(0,j),L})$. Assumption (16) and the Tchebyshev inequality imply that

$$\begin{aligned}\Sigma_2 &\geqslant \sum_{j=[\delta L^\gamma]+1}^{J'-1} \ln\left\{1 - \frac{I_{2,L}^{(d-1)}(\beta, \alpha - e_{(0,j),L})}{L^{d-1}\lambda^2[I_{1,L}^{(d-1)}(\beta, \alpha - e_{(0,j),L})]^2}\right\}\\ &\geqslant \frac{-2\ln 2}{L^{d-2}\lambda^2} \sum_{j=[\delta L^\gamma]+1}^{J'-1} \frac{1}{L} \frac{I_{2,L}^{(d-1)}(\beta, \alpha - e_{(0,j),L})}{[I_{1,L}^{(d-1)}(\beta, \alpha - e_{(0,j),L})]^2}\end{aligned} \tag{22}$$

Now we fix a value $E_0 > (\hbar^2\pi^2/2m)\,\delta^2 L^{2\gamma-2}$ in such a way that, for $E > E_0$,

$$\frac{I_{2,L}^{(d-1)}(\beta, \alpha - E)}{[I_{1,L}^{(d-1)}(\beta, \alpha - E)]^2} \leqslant \frac{3}{2} c^{1-d} \exp[\beta(E-\alpha)]$$

(with c as above), and split the sum in (22) into two parts, where $e_{(0,j),L}$ is greater (resp. less) than or equal to E_0. In the $L \to \infty$ limit, both parts tend to Riemann integrals; the first is finite for $\alpha < 0$ [with an upper bound of $O(L^{1-\gamma})$ independent of α]; the second one can be estimated using the inequality

$$\int_0^t e^{\beta y^2}\, dy < \frac{1}{\beta t} e^{\beta t^2}$$

(cf. ref. 12), which gives a bound of $O(L^{d-1}(\ln L)^{-1/2})$. Thus we get

$$\Sigma_2 = O(L(\ln L)^{-1/2}) \tag{23}$$

Inserting (19)–(21) and (23) into (14), we obtain

$$p_L^a(\beta, \mu) \geqslant \frac{a}{2}\left[\left(\frac{\mu-\alpha}{a}\right)^2 - R_1 - R_2\right] + \alpha x + O(L^{\gamma-1}) + p_L^0(\beta, \alpha) \tag{24}$$

If $\mu < a\rho_c(\beta) = aI_1^{(d)}(\beta, 0)$, we choose α such that

$$\mu - \alpha = aI_1^{(d)}(\beta, \alpha) \tag{25}$$

and set $x = 0$, $\delta = 0$. This implies [using (12) and (17)]

$$\left| NL^{-d} - \frac{\mu - \alpha}{a}\right| \leqslant \lambda I_1^{(d)}(\beta, \alpha) + O(L^{\gamma-1}\ln L)$$

and hence

$$\lim_{L\to\infty} R_1 \leqslant \lambda^2[\rho_c(\beta)]^2$$

From (13), we conclude that

$$\lim_{L\to\infty} R_2 = 0 \tag{26}$$

Since λ can be chosen arbitrarily small, we obtain

$$\lim_{L\to\infty} \inf p_L^a(\beta, \mu) \geqslant p^0(\beta, \alpha) + \frac{(\mu-\alpha)^2}{2a}$$

and, together with (6),

$$p^a(\beta, \mu) = p^0(\beta, \alpha) + \frac{(\mu-\alpha)^2}{2a}$$

where α satisfies (25).

Now suppose $\mu \geqslant a\rho_c(\beta)$. We choose some $\delta > 0$, $\alpha < 0$ and set

$$x = \mu/a - \rho_c(\beta)$$

From (12) and (17) we get

$$\left| NL^{-d} - \frac{\mu-\alpha}{a} \right| \leqslant \lambda\rho_c(\beta) - \frac{\alpha}{a} + O(L^{\gamma-1} \ln L)$$

and hence

$$\lim_{L\to\infty} R_1 \leqslant \left[\lambda\rho_c(\beta) - \frac{\alpha}{a} \right]^2$$

Furthermore, (26) remains true. Choosing now λ and $-\alpha$ arbitrarily small, we obtain from (24) and (6)

$$p^a(\beta, \mu) = p^0(\beta, 0) + \frac{\mu^2}{2a}$$

which coincides with the pressure of the imperfect Bose gas outlined in refs. 3 and 4.

This implies that such quantities as the mean energy density or the mean kinetic energy density will also coincide with the corresponding values for the mean-field model. Since the kinetic energy density attains its maximal value (for given β) at the critical density $\rho_c(\beta)$, while the total energy density continues to increase with increasing mean particle density, the further growth of the total energy density is caused by a macroscopic amount of particles with almost zero kinetic energy, which do not contribute to the mean kinetic energy density, i.e., there occurs Bose condensation in the generalized sense, as defined in ref. 10. On the other hand, macroscopic occupation of any single-particle state would give rise [cf. (4)] to a macroscopic variation of the mean interaction energy density with respect to the corresponding value for the mean-field model, hence condensation cannot occur in the usual understanding of the macroscopic occupation of some single-particle state.

REFERENCES

1. E. B. Davies, The thermodynamic limit for an imperfect boson gas, *Commun. Math. Phys.* **28**:69–86 (1972).
2. M. Fannes and A. Verbeure, The condensed phase of the imperfect Bose gas, *J. Math. Phys.* **21**:1809 (1980).
3. M. van den Berg, J. T. Lewis, and Ph. de Smedt, Condensation in the imperfect boson gas, *J. Stat. Phys.* **37**:697–707 (1984).
4. M. van den Berg, J. T. Lewis, and J. V. Pulè, The large deviations principle and some models of an interacting boson gas, *Commun. Math. Phys.* **118**:61–85 (1988).
5. M. van den Berg, J. T. Lewis, and J. V. Pulè, Large deviations and the boson gas, DIAS-STP-87-28.
6. J. T. Lewis, The large deviations principle in statistical mechanics: An expository account, DIAS-STP-87-26.
7. J. T. Lewis, Why do bosons condense?, DIAS-STP-86-12.
8. J. T. Lewis, J. V. Pulè, and Ph. de Smedt, The persistence of boson condensation in the van der Waals limit, preprint, Katholieke Universiteit Leuven.
9. Ph. de Smedt and V. A. Zagrebnov, Van der Waals limit of an interacting Bose gas in a weak external field, *Phys. Rev. A* **35**:4763–4769 (1987).
10. M. van den Berg, J. T. Lewis, and J. V. Pulè, A general theory of Bose–Einstein condensation, *Helv. Phys. Acta* **59**:1271–1288 (1986).
11. D. W. Robinson, Bose–Einstein condensation with attractive boundary conditions, *Commun. Math. Phys.* **50**:53 (1976).
12. M. Abramowitz, I. A. Stegun, *et al.*, *Handbook of Mathematical Functions* (NBS Applied Series, New York, 1964).